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Collinear permeable cracks between dissimilar piezoelectric materials

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Abstract

In this paper, the generalized two-dimensional problem of collinear interfacial cracks, between two dissimilar piezoelectric media subjected to piecewise uniform loads at infinity, is studied by means of the Stroh formalism. It is different from the relevant analysis done by other authors that in the present work, cracks are considered to be traction-free, but permeable slits across which both the normal component of the electric displacement and the tangential component of the electric field are continuous, and thus avoiding the common assumption of electric impermeability. According to the above continuous conditions combined with the principle of analytical continuation, the considered problem is reduced to a Hilbert problem. Explicit, closed-form expressions for the electric field inside cracks, complex potentials in piezoelectric media and field intensity factors near the crack tips are obtained. These results show that the electric field inside cracks is dependent on the material constants and the applied loads. It is also shown that all the field singularities are dependent only on the applied mechanical loads, not on the applied electric loads, which is different from those results based on impermeable crack model. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Piezoelectric; Interface; Permeable crack; Electric field

1. Introduction

With increasingly wide application of piezoelectric composite materials in the engineering, the study on the interfacial crack problem in piezoelectric media has received much interest. Suo et al. (1992) analyzed the generalized two-dimensional problem of collinear interfacial cracks between two dissimilar piezoelectric media in terms of Stroh formalism, and gave the structure of singular fields near the crack tips. However, it should be noted that their studies are based on the impermeable crack model, i.e., a

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crack is assumed to be a thin cut with impermeable faces, and thus the electric field inside the crack is neglected. This is the so-called impermeable crack assumption, which was widely used to investigate the crack problem in piezoelectric media to simplify analysis, see, for example, the work of Pak (1990, 1992), Sosa and Pak (1990), Sosa (1992), Wang (1992), Park and Sun (1995), Gao and Barnett (1996), Qin and Yu (1997), Zhong and Meguid (1997a, 1997b). In fact, cracks in the engineering are usually filled with air or vacuum, and the electric field inside cracks is a non-zero unknown quantity. Hence, as pointed out by McMeeking (1989), Pak and Tobin (1993), Dunn (1994), Sosa and Khutoryansky (1996), Kogan et al. (1996), Zhang et al. (1998), Gao and Fan (1998, 1999a), the impermeable crack assumption will lead to erroneous results. Recently, Beom and Atluri (1996) further addressed the interfacial crack problem in piezoelectric materials. They introduced a matrix function with which the intensity factors can be easily determined without solving the complicated eigenvalue problem. But it should be seen that Beom and Atluri's work is also based on the impermeable crack assumption. More recently, Shen and Kuang (1998) investigated the piezothermoelastic problem of collinear interfacial cracks between two half-infinite piezoelectric solids. In their study, the media are assumed to be subject to known loads of traction and electric displacement only along the crack faces, and as a result the crack-tip fields are obtained. However, in the engineering, one is often more interested in the full domain solutions of piezoelectric media loaded remotely. It is well known that the case of remote loading can be readily reduced to the case of crack-face loading by using the superposition principle, if the considered media are purely elastic. But it is not easy to do this if the media are piezoelectric. When the piezoelectric medium with a crack is loaded at infinity, the electric field exists inside the crack, and therefore the crack has to be considered as an 'electric inclusion'. This means that one has to overcome the difficulty to determine the electric field inside the crack. Otherwise, the electric displacement on the crack faces can not be known.

Similar to the crack problems in piezoelectric media, the problem of rigid line inclusions (sometimes called as hard cracks or inverse cracks) has also received much attention. For example, recently, Deng and Meguid (1998) addressed the generalized two-dimensional problem of an interfacial rigid line inclusion at the interface of two dissimilar piezoelectric materials. In their analysis, the inclusion is assumed as a conductor and thus the analysis process is simplified, since the electric field inside the inclusion can be considered to be zero. In fact, the interfacial inclusion, which results during electric packaging and manufacturing of intelligent composites, is usually a dielectric (Gao and Fan, 1999b).

It is the purpose of this study to investigate the generalized two-dimensional problem of interfacial cracks between two dissimilar piezoelectric half-spaces. One of the novel features in this paper is that the cracks are treated as permeable thin cuts across which both the normal component of the electric displacement and the tangential component of the electric field are continuous. The other novel feature is that the piezoelectric media are loaded at infinity, other than along the interfaces. The whole content consists of five sections. Following this brief introduction, Section 2 outlines the Stroh formalism to be needed in this paper, and then for two cases, we give the expressions for the electric field inside cracks, complex potentials in piezoelectric media and field intensity factors near the crack-tips, respectively, in Sections 3 and 4. The conclusions on the present work are drawn out in Section 5. In addition, the loading condition at infinity is formulated in Appendix A.

2. Stroh formalism

Consider a piezoelectric solid in a Cartesian system x_j ($j = 1, 2, 3$). Assuming that the displacement u_j and electric potential φ of the solid are dependent on x_1 and x_2 only, then the general solution for the generalized two-dimensional problem can be expressed as (Suo et al., 1992):

$$\mathbf{u} = \mathbf{A}\mathbf{f}(z) + \overline{\mathbf{A}\mathbf{f}(z)} \tag{1}$$

$$\phi = \mathbf{B}\mathbf{f}(z) + \overline{\mathbf{B}\mathbf{f}(z)} \tag{2}$$

with

$$\mathbf{u} = [u_1, u_2, u_3, \varphi]^{\mathrm{T}}, \quad \phi = [\phi_1, \phi_2, \phi_3, \phi_4]^{\mathrm{T}}$$

$$\mathbf{f}(z) = [f_1(z_1), f_2(z_2), f_3(z_3), f_4(z_4)]^{\mathrm{T}}, \quad z_k = x_1 + p_k x_2, \quad (k = 1, \ldots, 4)$$

In the above equations, the superscript T represents the transpose; the overbar stands for the conjugate of a complex number; $\mathbf{A}$ and $\mathbf{B}$ are two 4×4 matrices which can be determined from the material constants; $f_k(z_k)$ are complex potentials to be found; $\mathbf{u}$ and ϕ denote generalized displacement function vector and stress function vector, respectively; p_k $(k = 1, \ldots, 4)$ are the complex eigenvalues with positive imaginary parts; In this paper we assume that p_k are distinct. For this case, $\mathbf{A}$ and $\mathbf{B}$ are nonsingular, and there is the following orthogonality relation (Chung and Ting, 1996):

$$\begin{bmatrix} \mathbf{B}^T & \mathbf{A}^T \\ \bar{\mathbf{B}}^T & \bar{\mathbf{A}}^T \end{bmatrix} \begin{bmatrix} \mathbf{A} & \bar{\mathbf{A}} \\ \mathbf{B} & \bar{\mathbf{B}} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{3}$$

where $\mathbf{I}$ is the 4×4 unit matrix.

Once $\mathbf{f}(z)$ is obtained according to the given boundary conditions, the stress σ_{jl}, electric displacement D_l and electric field E_l can be given, respectively, by

$$\sigma_{j1} = -\phi_{j,2}, \sigma_{j2} = \phi_{j,1}, \quad (j = 1, 2, 3) \tag{4}$$

$$D_1 = -\phi_{4,2}, D_2 = \phi_{4,1}, E_1 = -u_{4,1}, E_2 = -u_{4,2} \tag{5}$$

where a comma indicates partial differentiation. It should be noted that the one-complex-variable approach introduced by Suo (1990) is used in this paper. After the solution of $\mathbf{f}(z)$ is obtained, one should substitute z_1, z_2, z_3 or z_4 for each component function of $\mathbf{f}(z)$ to calculate field quantities.

3. Interface cracks: non-oscillatory fields

Consider two dissimilar piezoelectric solids, one located in the upper half space V_1, and the other in the lower half space V_2, as shown in Fig. 1. The N interface cracks $l_n = \overline{a_n b_n}$ $(n = 1, 2, \ldots N)$ lie on the real axis x_1, and the union of the cracks and uncracked part in the x_1-axis are denoted by L_c and L_b, respectively. Moreover, it is assumed that the cracks are traction-free, but permeable slits filled with air or vacuum, while the upper and lower half-spaces are subjected to piecewise uniform loads at infinity and coexist in the state of generalized two dimensional deformation. The relation between the remote loads is given in Appendix A of this paper.

For this problem, the boundary conditions on the crack faces can be written as

$$\sigma_{2j}^+ = \sigma_{2j}^- = 0 \quad (j = 1, 2, 3) \quad \text{on } L_c \tag{6}$$

$$D_2^+ = D_2^-, E_1^+ = E_1^- \quad \text{on } L_c \tag{7}$$

On the bonded part, the continuous condition requires

$$\sigma_{2j}^{+} = \sigma_{2j}^{-},\ u_j^{+} = u_j^{-}, \quad (j = 1, 2, 3) \quad \text{on } \mathrm{L_b} \tag{8}$$

$$D_2^{+} = D_2^{-},\ E_1^{+} = E_1^{-} \quad \text{on } \mathrm{L_b} \tag{9}$$

Summarizing Eqs. (6)–(9), the above boundary conditions can be rearranged as

$$\sigma_{2j}^{+} = 0, \quad (j = 1, 2, 3) \quad \text{on } \mathrm{L_c} \tag{10}$$

$$\sigma_{2j}^{+} = \sigma_{2j}^{-},\ D_2^{+} = D_2^{-}, \quad -\infty < x_1 < \infty \tag{11}$$

$$E_1^{+} = E_1^{-}, \quad -\infty < x_1 < \infty \tag{12}$$

$$u_{j,\,1}^{+} = u_{j,\,1}^{-} \quad \text{on } \mathrm{L_b} \tag{13}$$

where $u_{j,\,1} = \partial u_j/\partial x_1$. The main task of the present work is to determine the complex potentials satisfying Eqs. (10)–(13).

For later use, define two vectors as

$$\phi_{,\,1} = (\sigma_{21}, \sigma_{22}, \sigma_{23}, D_2)^{\mathrm{T}},\ \mathbf{u}_{,\,1} = (u_{1,\,1}, u_{2,\,1}, u_{3,\,1}, u_{4,\,1})^{\mathrm{T}} \tag{14}$$

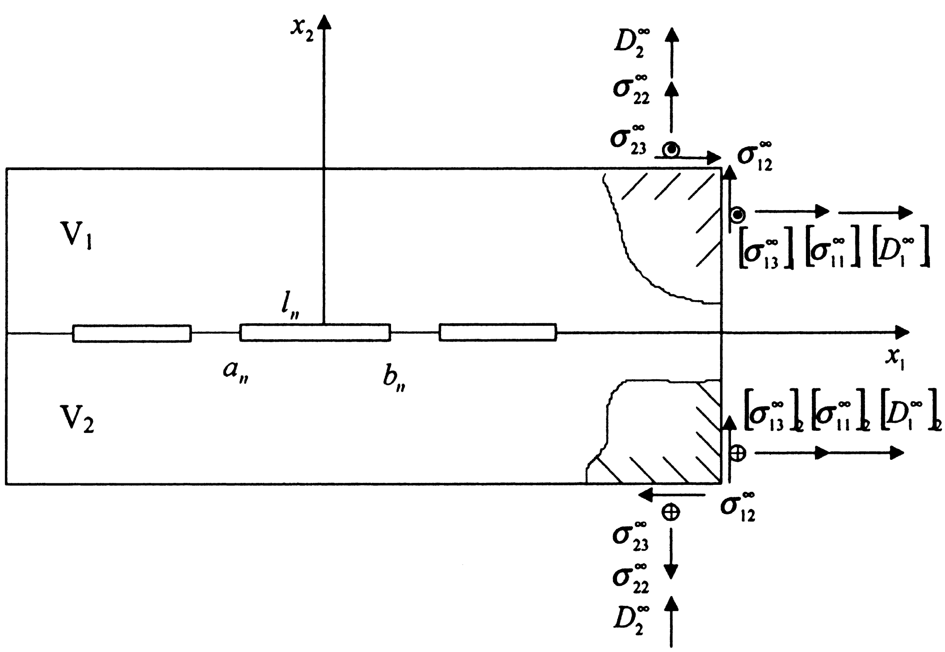

Fig. 1. Collinear cracks between two piezoelectric materials loaded at infinity.

where

$$u_{1,\,1} = \frac{\partial u_1}{\partial x_1} = \varepsilon_{11},\; u_{2,\,1} = \frac{\partial u_2}{\partial x_1} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) + \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) = \varepsilon_{12} + \omega_3,$$
$$u_{3,\,1} = \frac{\partial u_3}{\partial x_1} = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} = 2\varepsilon_{13},\; u_{4,\,1} = \frac{\partial u_4}{\partial x_1} = -E_1 \tag{15}$$

In Eq. (15), ε_{11}, ε_{12} and ε_{13} are strain components, respectively; ω_3 denotes rotation. On the other hand, one has from Eqs. (1) and (2) that

$$\boldsymbol{u}_{,\,1} = \mathbf{AF}(z) + \overline{\mathbf{AF}(z)} \tag{16}$$

$$\phi_{,\,1} = \mathbf{BF}(z) + \overline{\mathbf{BF}(z)} \tag{17}$$

where $\mathbf{F}(z) = \mathrm{d}\mathbf{f}(z)/\mathrm{d}z$.

For the considered problem, $\mathbf{F}(z)$ can be expressed, in the upper and lower spaces, as

$$\mathbf{F}_l(z) = \mathbf{C}_l^\infty + \mathbf{F}_{l0}(z) \quad (l = 1, 2) \tag{18}$$

where $\mathbf{F}_{l0}(z)$ is a function vector in V_1 ($l = 1$) or V_2 ($l = 2$), and $\mathbf{F}_{l0}(\infty) = \mathbf{0}$; $\mathbf{C}_l^\infty$ is a constant vector to be determined by the loading condition at infinity.

Substituting Eq. (18) into (16) and (17), and then take the limit $z \rightarrow \infty$ results in

$$\mathbf{A}_l\mathbf{C}_l^\infty + \bar{\mathbf{A}}_l\bar{\mathbf{C}}_l^\infty = \varepsilon_l^\infty \tag{19}$$

$$\mathbf{B}_l\mathbf{C}_l^\infty + \bar{\mathbf{B}}_l\bar{\mathbf{C}}_l^\infty = \sigma_l^\infty \tag{20}$$

where

$$\varepsilon_l^\infty = \left(\varepsilon_{11}^\infty, \varepsilon_{12}^\infty + \omega_3^\infty, 2\varepsilon_{13}^\infty, -E_1^\infty\right)_l^{\mathrm{T}} \tag{21}$$

$$\sigma_l^\infty = \left(\sigma_{21}^\infty, \sigma_{22}^\infty, \sigma_{23}^\infty, D_2^\infty\right)_l^{\mathrm{T}} \tag{22}$$

Eqs. (19) and (20) can be rewritten as

$$\begin{bmatrix} \mathbf{A} & \bar{\mathbf{A}} \\ \mathbf{B} & \bar{\mathbf{B}} \end{bmatrix}_l \begin{Bmatrix} \mathbf{C}_l^\infty \\ \bar{\mathbf{C}}_l^\infty \end{Bmatrix} = \begin{Bmatrix} \varepsilon_l^\infty \\ \sigma_l^\infty \end{Bmatrix} \tag{23}$$

Using Eq. (3), one has from Eq. (23) that

$$\begin{Bmatrix} \mathbf{C}_l^\infty \\ \bar{\mathbf{C}}_l^\infty \end{Bmatrix} = \begin{bmatrix} \mathbf{B}^{\mathrm{T}} & \mathbf{A}^{\mathrm{T}} \\ \bar{\mathbf{B}}^{\mathrm{T}} & \bar{\mathbf{A}}^{\mathrm{T}} \end{bmatrix}_l \begin{Bmatrix} \varepsilon_l^\infty \\ \sigma_l^\infty \end{Bmatrix} \tag{24}$$

Eq. (24) gives

$$\mathbf{C}_l^\infty = \mathbf{B}_l^{\mathrm{T}}\varepsilon_l^\infty + \mathbf{A}_l^{\mathrm{T}}\sigma_l^\infty \tag{25}$$

To determine the complete form of $\mathbf{F}_l(z)$, one has to use Eqs. (10)–(13). On x_1, (11) requires

$$\mathbf{B}_1\mathbf{F}_1(x_1) + \bar{B}_1\overline{F_1(x_1)} = \mathbf{B}_2\mathbf{F}_2(x_1) + \bar{\mathbf{B}}_2\overline{\mathbf{F}_2(x_1)}, \quad -\infty < x_1 < +\infty \tag{26}$$

Define a new analytical function as:

$$\mathbf{J}(z) = \begin{cases} \mathbf{B}_1\mathbf{F}_1(z) - \bar{\mathbf{B}}_2\overline{\mathbf{F}_2}(z), & z \in V_1 \\ \mathbf{B}_2\mathbf{F}_2(z) - \bar{\mathbf{B}}_1\overline{\mathbf{F}_1}(z), & z \in V_2 \end{cases} \tag{27}$$

Then, Eq. (26) can be reduced to

$$\mathbf{J}^+(x_1) - \mathbf{J}^-(x_1) = 0 \quad -\infty < x_1 < +\infty \tag{28}$$

The solution of Eq. (28) is given (Muskhelishvili, 1975) by

$$\mathbf{J}(z) = \mathbf{J}(\infty) = \mathbf{F}^\infty \tag{29}$$

where

$$\mathbf{F}^\infty = \mathbf{B}_1\mathbf{C}_1^\infty - \bar{\mathbf{B}}_2\bar{\mathbf{C}}_2^\infty \tag{30}$$

or

$$\mathbf{F}^\infty = \mathbf{B}_2\mathbf{C}_2^\infty - \bar{\mathbf{B}}_1\bar{\mathbf{C}}_1^\infty \tag{31}$$

Inserting Eq. (25) into (30) produces

$$\mathbf{F}^\infty = \mathbf{B}_1\mathbf{B}_1^{\mathrm{T}}\varepsilon_1^\infty - \mathbf{B}_2\mathbf{B}_2^{\mathrm{T}}\varepsilon_2^\infty + \left(\mathbf{B}_1\mathbf{A}_1^{\mathrm{T}} - \bar{\mathbf{B}}_2\bar{\mathbf{A}}_2^{\mathrm{T}}\right)\sigma_2^\infty \tag{32}$$

In addition, one can obtain from Eqs. (30) and (31) that

$$2\mathbf{F}^\infty = \left[\mathbf{B}_1\mathbf{C}_1^\infty - \overline{\mathbf{B}_1\mathbf{C}_1^\infty}\right] + \left[\mathbf{B}_2\mathbf{C}_2^\infty - \overline{\mathbf{B}_2\mathbf{C}_2^\infty}\right] \tag{33}$$

Eq. (33) shows that $\mathbf{F}^\infty$ is pure imaginary.

Eqs. (27) and (29) lead to

$$\mathbf{B}_1\mathbf{F}_1(z) - \bar{\mathbf{B}}_2\overline{\mathbf{F}_2}(z) = \mathbf{F}^\infty, \quad z \in V_1 \tag{34}$$

$$\mathbf{B}_2\mathbf{F}_2(z) - \bar{\mathbf{B}}_1\overline{\mathbf{F}_1}(z) = \mathbf{F}^\infty, \quad z \in V_2 \tag{35}$$

Introduce two auxiliary functions:

$$\Delta\mathbf{U}(x_1) = \mathbf{u}_{1,1}(x_1) - \mathbf{u}_{2,1}(x_1)$$

$$= \left[\mathbf{A}_1\mathbf{F}_1(x_1) + \bar{\mathbf{A}}_1\overline{\mathbf{F}_1(x_1)}\right] - \left[\mathbf{A}_2\mathbf{F}_2(x_1) + \bar{\mathbf{A}}_2\overline{\mathbf{F}_2(x_1)}\right] \tag{36}$$

$$\mathbf{T}(x_1) = \mathbf{B}_1\mathbf{F}_1(x_1) + \bar{\mathbf{B}}_1\overline{\mathbf{F}_1(x_1)} \tag{37}$$

Then, using Eqs. (34) and (35), Eq. (36) reduces to

$$i\Delta\mathbf{U}(x_1) = \mathbf{H}\left[\mathbf{B}_1\mathbf{F}_1(x_1) - \mathbf{H}^{-1}\bar{\mathbf{H}}\mathbf{B}_2\mathbf{F}_2(x_1) - \mathbf{H}^{-1}\left(\bar{\mathbf{Y}}_2 - \bar{\mathbf{Y}}_1\right)\mathbf{F}^\infty\right] \tag{38}$$

where

$$\mathbf{Y}_1 = i\mathbf{A}_1\mathbf{B}_1^{-1}, \quad \mathbf{Y}_2 = i\mathbf{A}_2\mathbf{B}_2^{-1}, \quad \mathbf{H} = \mathbf{Y}_1 + \bar{\mathbf{Y}}_2 \tag{39}$$

By defining

$$\mathbf{K}(z) = \begin{cases} \mathbf{B}_1\mathbf{F}_1(z) & z \in V_1 \\ \mathbf{H}^{-1}\bar{\mathbf{H}}\mathbf{B}_2\mathbf{F}_2(z) + \mathbf{H}^{-1}\left(\bar{\mathbf{Y}}_2 - \bar{\mathbf{Y}}_1\right)\mathbf{F}^{\infty}, & z \in V_2 \end{cases} \tag{40}$$

Eq. (38) can be expressed as

$$i\Delta\mathbf{U}(x_1) = \mathbf{H}\left[\mathbf{K}^+(x_1) - \mathbf{K}^-(x_1)\right] \tag{41}$$

Noting that (12) and (13) imply $\Delta\mathbf{U}(x_1) = 0$ on $\mathrm{L_b}$, and therefore, (41) shows that $\mathbf{K}(z)$ is analytic in the entire z-plane except on $\mathrm{L_c}$.

Moreover, the continuous condition of E_1 on the x_1-axis, i.e., (12), requires

$$\mathbf{H}_4\left[\mathbf{K}^+(x_1) - \mathbf{K}^-(x_1)\right] = \mathbf{0}, \quad -\infty < x_1 < \infty \tag{42}$$

where

$$\mathbf{H}_4 = (H_{41}, H_{42}, H_{43}, H_{44}) \tag{43}$$

The solution of (42) is

$$\mathbf{H}_4\mathbf{K}(z) = \mathbf{H}_4\mathbf{K}(\infty) \tag{44}$$

Eq. (44) gives

$$K_4(z) = -\frac{1}{H_{44}}\sum_{j=1}^{3} H_{4j}K_j(z) + \frac{1}{H_{44}}\mathbf{H}_4\mathbf{K}(\infty) \tag{45}$$

Similarly, by using Eqs. (34) and (35), one has from (37) that

$$\mathbf{T}(x_1) = \mathbf{K}^+(x_1) + \bar{\mathbf{H}}^{-1}\mathbf{H}\mathbf{K}^-(x_1) - \bar{\mathbf{H}}^{-1}\left(\mathbf{Y}_2 + \bar{\mathbf{Y}}_2\right)\mathbf{F}^{\infty}, \quad -\infty < x_1 < +\infty \tag{46}$$

On the bonded part, $\mathbf{K}(x_1)$ is analytic, and thus Eq. (46) becomes

$$\mathbf{T}(x_1) = \left(\mathbf{I} + \bar{\mathbf{H}}^{-1}\mathbf{H}\right)\mathbf{K}(x_1) - \mathbf{K}_0, \quad x_1 \in \mathrm{L_b} \tag{47}$$

where

$$\mathbf{K}_0 = \bar{\mathbf{H}}^{-1}\left(\mathbf{Y}_2 + \bar{\mathbf{Y}}_2\right)\mathbf{F}^{\infty}$$

On the crack faces, one has $\mathbf{T}(x_1) = \mathbf{i}_4 D_2(x_1)$, where $\mathbf{i}_4 = (0, 0, 0, 1)^{\mathrm{T}}$ and $D_2(x_1)$ is an unknown function which indicates the boundary value of $D_2(z)$ on the crack faces. Hence, Eq. (47) reduces

$$\mathbf{K}^+(x_1) + \bar{\mathbf{H}}^{-1}\mathbf{H}\mathbf{K}^-(x_1) = \mathbf{K}_0 + \mathbf{i}_4 D_2(x_1), \quad x_1 \in \mathrm{L_c} \tag{48}$$

Obviously once one obtains $\mathbf{K}(z)$ from Eqs. (48) and (44), $\mathbf{F}_1(z)$ and $\mathbf{F}_2(z)$ can be given by using (40), and then all the field solutions can be determined without difficulty.

First let us examine a special case when $\mathbf{H}$ is real. For this case (48) degenerates into

$$\mathbf{K}^{+}(x_1) + \mathbf{K}^{-}(x_1) = \mathbf{K}_0 + \mathbf{i}_4 D_2(x_1), \quad x_1 \in \mathrm{L_c} \tag{49}$$

in which $\mathbf{K}_0$ is a pure imaginary vector.

Following Muskhelishvili (1975) and noting that $D_2(z)$ is a bounded function at infinity (i.e., $D_2(\infty) = D_2^{\infty}$), it can be shown that the solution of Eq. (49) is

$$\mathbf{K}(z) = \frac{1}{2}\left[\mathbf{K}_0 + \mathbf{i}_4 D_2(z)\right] + X(z)\mathbf{P}(z) \tag{50}$$

where

$$X(z) = \prod_{n=1}^{N} (z - a_n)^{-\frac{1}{2}} (z - b_n)^{-\frac{1}{2}} \tag{51}$$

$$\mathbf{P}(z) = \mathbf{c}_N z^N + \mathbf{c}_{N-1} z^{N-1} + \cdots + \mathbf{c}_0 \tag{52}$$

$$\mathbf{c}_n = \left[c_n^{(1)}, c_n^{(2)}, c_n^{(3)}, c_n^{(4)}\right]^{\mathrm{T}} \quad (n = N, N-1, \ldots, 0) \tag{53}$$

In Eq. (50), $D_2(z)$ and the constant vectors $\mathbf{c}_n$ contained in $\mathbf{P}(z)$ are unknown. To find $D_2(z)$, inserting (50) into (44) produces

$$D_2(z) = \frac{1}{H_{44}} \mathbf{H}_4 \left[2\mathbf{K}(\infty) - \mathbf{K}_0\right] - 2X(z)\mathbf{H}_4\mathbf{P}(z) \tag{54}$$

On the other hand, letting $x_1 \to \infty$ in (47) leads to

$$2\mathbf{K}(\infty) - \mathbf{K}_0 = \sigma_2^{\infty} \tag{55}$$

where

$$\sigma_2^{\infty} = \left(\sigma_{21}^{\infty}, \sigma_{22}^{\infty}, \sigma_{23}^{\infty}, D_2^{\infty}\right)^{\mathrm{T}}$$

Substituting (55) into (54) gives

$$D_2(z) = D_2^{\infty} + \frac{1}{H_{44}} \sum_{j=1}^{3} H_{4j} \sigma_{2j}^{\infty} - 2X(z)\mathbf{H}_4\mathbf{P}(z) \tag{56}$$

Furthermore, let us determine $\mathbf{c}_n$. Taking the limit $z \to \infty$ in (50) yields

$$\mathbf{K}(\infty) = \frac{1}{2}\left[\mathbf{K}_0 + \mathbf{i}_4 D_2^{\infty}\right] + \mathbf{c}_N \tag{57}$$

Substituting Eq. (57) into (55) leads to

$$\mathbf{c}_N = \frac{1}{2}\sigma_M^{\infty} \tag{58}$$

where σ_M^{∞} is a constant vector dependent only on the mechanical loads at infinity, such that

$$\sigma_M^{\infty} = \left(\sigma_{21}^{\infty}, \sigma_{22}^{\infty}, \sigma_{23}^{\infty}, 0\right)^{\mathrm{T}} \tag{59}$$

To find the remaining unknown constants $\mathbf{c}_n$, ($n = (N-1,\ldots,0)$ in Eq. (52), one must use the single-valued conditions of displacement and electric potential. Considering (41), these conditions require

$$\oint_{\Gamma_n} \mathbf{K}(z)\,\mathrm{d}z = 0 \quad (1, 2, \ldots, N) \tag{60}$$

where Γ_n is a clockwise closed-contour encircled the crack l_n.

Substituting (50) into (60), and then using $\oint_{\Gamma_n} D_2(z)\,\mathrm{d}z = 0$ (since the total net charge on the cracks is zero), one obtains

$$\oint_{\Gamma_n} X(z)\mathbf{P}(z)\,\mathrm{d}z = 0 \tag{61}$$

It can be found, from Eq. (61) together with Eq. (58), that all the coefficients in Eq. (52) are real, and they depend only on the applied mechanical loads and the size of cracks, but on the applied electric loads.

On the other hand, to determine the electric field inside cracks, letting $z = x_1^+$ and $z = x_1^-$ in Eq. (56), respectively, one has

$$D_2^0\left(x_1^+\right) = D_2^\infty + \frac{1}{H_{44}}\sum_{j=1}^{3} H_{4j}\sigma_{2j}^\infty - 2X\left(x_1^+\right)\mathbf{H}_4\mathbf{P}\left(x_1^+\right) \tag{62}$$

$$D_2^0(x_1^-) = D_2^\infty + \frac{1}{H_{44}}\sum_{j=1}^{3} H_{4j}\sigma_{2j}^\infty - 2X(x_1^-)\mathbf{H}_4\mathbf{P}(x_1^-) \tag{63}$$

Since $D_2^0(x_1^+) = D_2^0(x_1^-)$, $\mathbf{P}(x_1^+) = \mathbf{P}(x_1^-)$ and $X(x_1^+) = -X(x_1^-)$ on the crack faces, Eqs. (62) and (63) leads to

$$D_2^0(x_1) = D_2^\infty + \frac{1}{H_{44}}\sum_{j=1}^{3} H_{4j}\sigma_{2j}^\infty \tag{64}$$

Eq. (64) shows that D_2^0 equals a constant inside any crack.

Using Eq. (64), the component of electric field in the x_2-axis direction can be expressed as

$$E_2^0 = \frac{D_2^\infty}{\varepsilon_0} + \frac{1}{\varepsilon_0 H_{44}}\sum_{j=1}^{3} H_{4j}\sigma_{2j}^\infty \tag{65}$$

where ε_0 is the dielectric constant of air.

After the above unknowns are determined, one can obtain the complete solutions of the complex potentials, and then can finally give the expression of $E_1(z)$ at arbitrary position, including that on the crack face. Thus, $E_1^0(x_1)$ can be determined. It can be shown that $E_1^0(x_1)$ is in general not constant inside the crack, and it exhibits the classical square root singularity as x_1 approaches the crack tips from on the crack faces.

At the right tip of any crack l_n, the field intensity factor vector $\mathbf{k}(b_n)$ can be expressed as

$$\mathbf{k}(b_n) = \lim_{x_1 \to b_n} \sqrt{2\pi}(x_1 - b_n)^{1/2}\mathbf{T}(x_1) \tag{66}$$

Substituting (47) into (66) results in

$$\mathbf{k}(b_n) = 2\sqrt{2\pi} \lim_{x_1 \to b_n} (x_1 - b_n)^{1/2} \mathbf{K}(x_1) \tag{67}$$

From (67), the stress intensity factor $k_j(b_n)$ ($j = 1, 2, 3$) and the intensity factor of electric displacement $k_4(b_n)$ can be expressed, respectively, as

$$k_j(b_n) = 2\sqrt{2\pi} \lim_{x_1 \to b_n} (x_1 - b_n)^{1/2} K_j(x_1), \quad (j = 1, 2, 3) \tag{68}$$

$$k_4(b_n) = 2\sqrt{2\pi} \lim_{x_1 \to b_n} (x_1 - b_n)^{1/2} K_4(x_1) \tag{69}$$

Inserting Eqs. (50) and (45) into Eqs. (68) and (69) gives

$$k_j(b_n) = 2\sqrt{2\pi} \lim_{x_1 \to b_n} (x_1 - b_n)^{1/2} X(x_1) P^{(j)}(x_1) \quad (j = 1, 2, 3) \tag{70}$$

$$k_4(b_n) = -\frac{1}{H_{44}} - \sum_{j=1}^{3} H_{4j} k_j(b_n) \tag{71}$$

Eq. (71) shows that the singularity of electric displacement depends on that of the stresses, while Eq. (70) together with (52) indicates that the stress intensity factor is related to $\mathbf{c}_n$ ($n = N, N-1, \ldots, 0$). One can find from Eqs. (61) and (58) that $\mathbf{c}_n$ are dependent on the applied mechanical loads and crack size, but not on the applied electric loads and the material constants. This means that the stress intensity factor is identical to that of the corresponding isotropic material. Thus, Eqs. (70) and (71) imply also that the applied electric loads have no influence on all the field singularities.

For the case of a homogeneous piezoelectric medium, when D_2^∞ is applied solely at infinity, Eqs. (58), (61) and (56) provide $\mathbf{c}_n = \mathbf{0}$ and $D_2(z) = D_2^\infty$, and therefore (50) becomes

$$\mathbf{K}(z) = \frac{1}{2}\left[\mathbf{K}_0 + \mathbf{i}_4 D_2^\infty\right] \tag{72}$$

Since $\mathbf{K}_0$ is pure imaginary vector, Eq. (72) indicates that in this case, the stress in the piezoelectric medium is zero, while $D_2 = D_2^0 = D_2^\infty$. Similar results are also found by Kogan et al. (1996) and Gao and Fan (1999a), who respectively studied the problems of a penny-shaped crack and a straight-line crack in a transversely isotropic material using exact boundary conditions. On the other hand, if taking a dielectric medium as a special case of a piezoelectric medium, one can immediately write the corresponding solutions of the dielectric medium with cracks according to the above result. For a case of a crack in the dielectric material subjected to uniform electric loads at infinity, it can be confirmed that the result produced from the current work is the same as that of Wangsness (1979). However, for this case if the impermeable assumption is used, one can find, from the work of Suo et al. (1992) and Park and Sun (1995), that D_2 is singular near the crack tips.

4. Interface cracks: oscillatory fields

In this section we examine the general case when $\mathbf{H}$ is complex. For the sake of convenience, it is assumed that eigenvalues of $\bar{\mathbf{H}}^{-1}\mathbf{H}$ is of the form $-e^{-2\pi i \delta_\alpha}$

$$\delta_\alpha = -\frac{1}{2} + i\varepsilon_\alpha \tag{73}$$

According to the definition of eigenvalues, one has

$$\| - \mathrm{e}^{-2\pi i\delta_\alpha}\mathbf{I} - \bar{\mathbf{H}}^{-1}\mathbf{H}\| = 0 \tag{74}$$

Noting

$$-2\pi i\delta_\alpha = -2\pi i\left(-\frac{1}{2} + i\varepsilon_\alpha\right) = \pi i + 2\pi\varepsilon_\alpha \tag{75}$$

we have

$$-\mathrm{e}^{-2\pi i\delta_\alpha} = \mathrm{e}^{2\pi\varepsilon_\alpha} \tag{76}$$

Thus, Eq. (74) reduces to

$$\|\mathbf{H} - \mathrm{e}^{2\pi\varepsilon_\alpha}\bar{\mathbf{H}}\| = 0 \tag{77}$$

Following the work of Suo et al. (1992), (77) results in

$$\varepsilon_1 = \varepsilon,\ \varepsilon_2 = -\varepsilon,\ \varepsilon_3 = -i\kappa,\ \varepsilon_4 = i\kappa \tag{78}$$

where

$$\varepsilon = \frac{1}{\pi}\tanh^{-1}\left[(b^2 - c^2)^{1/2} - b\right]^{1/2}$$

$$\kappa = \frac{1}{\pi}\tanh^{-1}\left[(b^2 - c^2)^{\frac{1}{2}} + b\right]^{1/2}$$

$$b = \frac{1}{4}\mathrm{tr}\left[(\mathbf{D}^{-1}\mathbf{W})^2\right]$$

$$c = \|\mathbf{D}^{-1}\mathbf{W}\|$$

$$\mathbf{D} + i\mathbf{W} = \mathbf{H}$$

Letting $\mathbf{Q}$ be the eigenvector matrix of $\bar{\mathbf{H}}^{-1}\mathbf{H}$, one has

$$\mathbf{Q}^{-1}\bar{\mathbf{H}}^{-1}\mathbf{H}\mathbf{Q} = \Lambda, \quad \Lambda = \langle\langle -\mathrm{e}^{-2\pi i\delta_\alpha}\rangle\rangle = \langle\langle \mathrm{e}^{2\pi\varepsilon_\alpha}\rangle\rangle \tag{79}$$

where the angular $\langle\langle\ \rangle\rangle$ indicates the diagonal matrix in which each component is varied according to the Greek index α.

Multiplying both sides of (48) by $\mathbf{Q}^{-1}$ leads to

$$\left[\mathbf{Q}^{-1}\mathbf{K}(x_1)\right]^+ + \mathbf{Q}^{-1}\bar{\mathbf{H}}^{-1}\mathbf{H}\mathbf{Q}\left[\mathbf{Q}^{-1}\mathbf{K}(x_1)\right]^- = \mathbf{Q}^{-1}\left[\mathbf{K}_0 + \mathbf{i}_4 D_2(x_1)\right] \tag{80}$$

Letting

$$\mathbf{Q}^{-1}\mathbf{K}(z) = \mathbf{R}(z) \tag{81}$$

$$\mathbf{Q}^{-1}\left[\mathbf{K}_0 + \mathbf{i}_4 D_2(x_1)\right] = \mathbf{R}^0(z) \tag{82}$$

and considering Eq. (79), Eq. (80) can be reduced to

$$\mathbf{R}^+(x_1) + \Lambda\mathbf{R}^-(x_1) = \mathbf{R}^0(x_1) \tag{83}$$

Expanding (83), one obtains

$$R_\alpha^+(x_1) - \mathrm{e}^{-2\pi i\delta_\alpha} R_\alpha^-(x_1) = R_\alpha^0(x_1) \quad (\alpha = 1,\ldots,4) \tag{84}$$

Following Muskhelishvili (1975), the solution of (84) is given by

$$R_\alpha(z) = \frac{X_0^\alpha(z)}{2\pi i}\int_L \frac{R_\alpha^0(x_1)\mathrm{d}x_1}{X_0^\alpha(x_1)(x_1 - z)} + X_0^\alpha(z)P_\alpha^*(z) \tag{85}$$

where $P_\alpha^*(z)$ is an N degree polynomial, and

$$X_0^\alpha(z) = \prod_{j=1}^{N}(z - a_j)^{-\gamma_\alpha}(z - a_j)^{\gamma_\alpha - 1} \tag{86}$$

$$\gamma_\alpha = \frac{1}{2\pi i}\ln g_\alpha, \quad g_\alpha = \mathrm{e}^{-2\pi i\delta_\alpha} \tag{87}$$

Using the following identities:

$$\gamma_\alpha = -\delta_\alpha,\ \delta_\alpha = -\frac{1}{2} + i\varepsilon_\alpha,\ 1 + \delta_\alpha = \frac{1}{2} + i\varepsilon_\alpha \tag{88}$$

Eq. (86) can be rewritten as

$$X_0^\alpha(z) = \prod_{j=1}^{N}\frac{\left(\dfrac{z - a_j}{z - b_j}\right)^{i\varepsilon_\alpha}}{\sqrt{(z - a_j)(z - b_j)}} \tag{89}$$

It can be shown that (85) can be finally reduced to

$$R_\alpha(z) = \frac{1}{1 + \mathrm{e}^{2\pi\varepsilon_\alpha}}R_0^\alpha(z) + X_0^\alpha(z)P_\alpha(z) \tag{90}$$

where $P_\alpha(z)$ is a new N polynomial:

$$P_\alpha(z) = c_N^{(\alpha)}z^N + c_{N-1}^{(\alpha)}z^{N-1} + \cdots + c_0^{(\alpha)} \tag{91}$$

On the other hand, (90) can be rewritten in the vector form as

$$\mathbf{R}(z) = \langle\langle\frac{1}{1 + \mathrm{e}^{2\pi\varepsilon_\alpha}}\rangle\rangle\mathbf{R}_0(z) + \langle\langle X_0^\alpha(z)\rangle\rangle\mathbf{P}(z) \tag{92}$$

where $\mathbf{P}(z)$ is given by (52).

Considering Eqs. (81) and (82), Eq. (92) gives

$$\mathbf{K}(z) = \mathbf{Q}\langle\langle\frac{1}{1+\mathrm{e}^{2\pi\varepsilon_\alpha}}\rangle\rangle\mathbf{Q}^{-1}\left[\mathbf{K}_0 + \mathbf{i}_4 D_2(z)\right] + \mathbf{Q}\langle\langle X_0^\alpha(z)\rangle\rangle\mathbf{P}(z) \tag{93}$$

To find the expression of $D_2(z)$ in (93), inserting (93) into (44) yields

$$D_2(z) = \frac{1}{C_D}\mathbf{H}_4\left[\mathbf{K}(\infty) - \mathbf{Q}\langle\langle\frac{1}{1+\mathrm{e}^{2\pi\varepsilon_\alpha}}\rangle\rangle\mathbf{Q}^{-1}\mathbf{K}_0\right] - \frac{1}{C_D}\mathbf{H}_4\mathbf{Q}\langle\langle X_0^\alpha(z)\rangle\rangle\mathbf{P}(z) \tag{94}$$

where

$$C_D = \mathbf{H}_4\mathbf{Q}\langle\langle\frac{1}{1+\mathrm{e}^{2\pi\varepsilon_\alpha}}\rangle\rangle\mathbf{Q}^{-1}\mathbf{i}_4 \tag{95}$$

Taking the limit as $x_1 \to \infty$ in Eq. (47) produces

$$\left(\mathbf{I} + \bar{\mathbf{H}}^{-1}\mathbf{H}\right)\mathbf{K}(\infty) - \mathbf{K}_0 = \sigma_2^\infty \tag{96}$$

Multiplying both sides of Eq. (96) by $\mathbf{Q}^{-1}$ leads to

$$\mathbf{Q}^{-1}\left(\mathbf{I} + \bar{\mathbf{H}}^{-1}\mathbf{H}\right)\mathbf{Q}\mathbf{Q}^{-1}\mathbf{K}(\infty) = \mathbf{Q}^{-1}\left(\mathbf{K}_0 + \sigma_2^\infty\right) \tag{97}$$

Using Eq. (79), Eq. (97) becomes

$$\langle\langle 1 + \mathrm{e}^{2\pi\varepsilon_\alpha}\rangle\rangle\mathbf{Q}^{-1}\mathbf{K}(\infty) = \mathbf{Q}^{-1}\left(\mathbf{K}_0 + \sigma_2^\infty\right) \tag{98}$$

From Eq. (98), one obtains

$$\mathbf{K}(\infty) = \mathbf{Q}\langle\langle\frac{1}{1+\mathrm{e}^{2\pi\varepsilon_\alpha}}\rangle\rangle\mathbf{Q}^{-1}\left(\mathbf{K}_0 + \sigma_2^\infty\right) \tag{99}$$

Inserting Eq. (99) into (94), we obtain the final expression of electric displacement as

$$D_2(z) = \frac{1}{C_D}\mathbf{H}_4\mathbf{Q}\langle\langle\frac{1}{1+\mathrm{e}^{2\pi\varepsilon_\alpha}}\rangle\rangle\mathbf{Q}^{-1}\sigma_2^\infty - \frac{1}{C_D}\mathbf{H}_4\mathbf{Q}\langle\langle X_0^\alpha(z)\rangle\rangle\mathbf{P}(z) \tag{100}$$

In addition, taking the limit as $z \to \infty$ in Eq. (93) yields

$$\mathbf{K}(\infty) = \mathbf{Q}\langle\langle\frac{1}{1+\mathrm{e}^{2\pi\varepsilon_\alpha}}\rangle\rangle\mathbf{Q}^{-1}\left[\mathbf{K}_0 + \mathbf{i}_4 D_2^\infty\right] + \mathbf{Q}\mathbf{c}_N \tag{101}$$

Inserting Eq. (99) into (101), we have

$$\mathbf{c}_N = \langle\langle\frac{1}{1+\mathrm{e}^{2\pi\varepsilon_\alpha}}\rangle\rangle\mathbf{Q}^{-1}\sigma_M^\infty \tag{102}$$

Eq. (102) together with (59) shows that $\mathbf{c}_N$ is still independent of the applied electric loads.

To find the remaining coefficients $\mathbf{c}_n$ ($n = N-1, N-2, \ldots, 0$), substituting Eq. (93) into (60) gives

$$\oint_{\Gamma_n} \mathbf{Q}\langle\langle X_0^\alpha(z)\rangle\rangle\mathbf{P}(z)\,\mathrm{d}z = 0 \tag{103}$$

Since $\mathbf{Q}$ is not singular, Eq. (103) can be simplified to

$$\oint_{\Gamma_n} \langle\langle X_0^{\alpha}(z)\rangle\rangle \mathbf{P}(z)\,\mathrm{d}z = 0 \tag{104}$$

Below let us study the electric field inside cracks. In Eq. (100), letting z equal x_1^+ and x_1^-, respectively, one has

$$D_2^0(x_1^+) = \frac{1}{C_D}\mathbf{H}_4\mathbf{Q}\langle\langle\frac{1}{1+\mathrm{e}^{2\pi\varepsilon_\alpha}}\rangle\rangle\mathbf{Q}^{-1}\sigma_2^\infty - \frac{1}{C_D}\mathbf{H}_4\mathbf{Q}\langle\langle X_0^\alpha(x_1^+)\rangle\rangle\mathbf{P}(x_1^+) \tag{105}$$

$$D_2^0(x_1^-) = \frac{1}{C_D}\mathbf{H}_4\mathbf{Q}\langle\langle\frac{1}{1+\mathrm{e}^{2\pi\varepsilon_\alpha}}\rangle\rangle\mathbf{Q}^{-1}\sigma_2^\infty - \frac{1}{C_D}\mathbf{H}_4\mathbf{Q}\langle\langle X_0^\alpha(x_1^-)\rangle\rangle\mathbf{P}(x_1^-) \tag{106}$$

Using the following relations

$$D_2^0(x_1^+) = D_2^0(x_1^-),\ \mathbf{P}(x_1^+) = \mathbf{P}(x_1^-),\ X_0^\alpha(x_1^-) = -\mathrm{e}^{-2\pi\varepsilon_\alpha}X_0^\alpha(x_1^+) \tag{107}$$

one obtains from Eqs. (105) and (106) that

$$D_2^0(x_1) = \frac{1}{C_D}\mathbf{H}_4\mathbf{Q}\langle\langle\frac{1}{1+\mathrm{e}^{2\pi\varepsilon_\alpha}}\rangle\rangle\mathbf{Q}^{-1}\sigma_2^\infty - \frac{1}{2C_D}\mathbf{H}_4\mathbf{Q}\langle\langle\frac{\mathrm{e}^{2\pi\varepsilon_\alpha}-1}{\mathrm{e}^{2\pi\varepsilon_\alpha}}\rangle\rangle\langle\langle X_0^\alpha(x_1^+)\rangle\rangle\mathbf{P}(x_1) \tag{108}$$

For the general cases, $\varepsilon_\alpha \neq 0$, and thus the second term in the right-hand-side of Eq. (108) is not equal to zero. This means that $D_2^0(x_1)$ varies inside cracks, and moreover it is singular and oscillatory as x_1 approaches the crack tips from on the crack faces. Similar nature can also be found for $E_1^0(x_1)$.

Substituting Eq. (93) together with (100) into (47), one can obtain the singular principal part of $\mathbf{T}(r)$ ahead of the crack tip ($x_{10} = b_n$). The result is

$$\mathbf{T}(r) = \mathbf{V}\langle\langle X_0^\alpha(x_1)\rangle\rangle\mathbf{P}(x_1) \tag{109}$$

where r means the distance from the crack-tip; $x_1 = b_n + r$; and

$$\mathbf{V} = \left(\mathbf{I} + \bar{\mathbf{H}}^{-1}\mathbf{H}\right)[\mathbf{Q}+\mathbf{R}]$$

$$R = -\frac{1}{C_D}\mathbf{Q}\langle\langle\frac{1}{1+\mathrm{e}^{2\pi\varepsilon_\alpha}}\rangle\rangle\mathbf{Q}^{-1}\mathbf{i}_4\mathbf{H}_4\mathbf{Q}$$

Thus, the field intensity factor vector may be defined as

$$\mathbf{k} = [k_{II}, k_I, k_{III}, k_D]^{\mathrm{T}} = \lim_{r\to 0}\sqrt{2\pi r}\,\mathbf{V}\langle\langle r^{i\varepsilon_\alpha}\rangle\rangle\langle\langle X_0^\alpha(x_1)\rangle\rangle\mathbf{P}(x_1) \tag{110}$$

Observing Eqs. (110) and (52), one can find that $\mathbf{k}$ is related to $\mathbf{c}_n$, while (102) and (103) show that $\mathbf{c}_n$ are independent of the applied electric loads. This implies that in general cases, the singularities of the stress and electric displacement depend only on the applied mechanical loads and the material constants.

As an example, consider a case of a crack which is located in $[-a, a]$. In this case, $a_1 = -a$, $b_1 = a$, and

$$X_0^\alpha(z) = \frac{1}{\sqrt{z^2-a^2}}\left(\frac{z+a}{z-a}\right)^{i\varepsilon_\alpha} \tag{111}$$

$$\mathbf{P}(z) = \mathbf{c}_N z \quad (N=1) \tag{112}$$

Substituting Eqs. (111) and (112) into (110), one obtains

$$\mathbf{k}_\sigma(a) = \sqrt{\pi a} V\langle\langle(2a)^{i\varepsilon_\alpha}\rangle\rangle \mathbf{Q}^{-1}\sigma_M^\infty \tag{113}$$

For the case of a homogeneous medium, one has

$$\varepsilon_\alpha = 0,\ \mathbf{Q} = \mathbf{I},\ C_D = \frac{1}{2}H_{44} \tag{114}$$

$$\mathbf{R} = -\frac{1}{H_{44}}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix},\quad \mathbf{V} = 2\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\dfrac{H_{41}}{H_{44}} & -\dfrac{H_{42}}{H_{44}} & -\dfrac{H_{43}}{H_{44}} & 0 \end{bmatrix} \tag{115}$$

Inserting Eqs. (114) and (115) into (113) gives

$$k_I = \sqrt{\pi a}\sigma_{22}^\infty,\ k_{II} = \sqrt{\pi a}\sigma_{21}^\infty,\ k_{III} = \sqrt{\pi a}\sigma_{23}^\infty$$

$$k_D = -\frac{1}{H_{44}}\sum_{j=1}^{3} H_{4j}\sigma_{2j}^\infty \tag{116}$$

which are consistent with those of Gao and Fan (1998), who analyzed an elliptic hole problem in piezoelectric media by use of the Stroh formalism and exact boundary conditions. However, if the impermeable crack assumption is used, one has $k_D = \sqrt{\pi a}D_2^\infty$ (e.g., Suo et al., 1992). This means that the impermeable crack assumption may lead to erroneous results for the crack problem in piezoelectric media.

5. Conclusions

Based on the Stroh formalism, a theoretical study is done on the two-dimensional problems of N collinear permeable cracks between two dissimilar piezoelectric solids subjected to uniform loads at infinity. Exact and explicit solutions are presented both in the solids and inside the cracks. From these results, several conclusions may be reached:

1. Near the permeable crack-tips, the structure of singular fields is the same as that near the impermeable crack-tips. However, for the case of a permeable crack all the field singularities are dependent on the material constants and the applied mechanical loads, but not on the applied electric loads.
2. In general cases, the electric field inside interfacial cracks depends on the material constants, the applied loads and dielectric constant of air. Moreover, it can be singular and oscillatory when approaching the crack tips. For the case of a homogeneous piezoelectric medium with cracks, the electric field inside any crack equals a constant.
3. All the field variables in piezoelectric media are independent of dielectric constant of air or vacuum inside cracks.
4. In order to ensure the deformation of the upper and lower half-space compatible, the loads applied at infinity has to satisfy certain condition, which is given in appendix.

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Appendix A. The condition of loading at infinity

Consider a piezoelectric solid in a fixed rectangular coordinate system x_j ($j = 1, 2, 3$). Taking stresses σ_{kl} and electric displacement D_k as independent variables, the constitutive equations can be expressed as (Berlincourt et al., 1964)

$$\varepsilon_{ij} = s_{ijkl}\sigma_{kl} + g_{kij}D_k, \ -E_i = g_{ikl}\sigma_{kl} - \beta_{ik}D_k \quad (i, k, l = 1, 2, 3) \tag{A1}$$

where s_{ijkl}, g_{kij} and β_{ik} are elastic constants, piezoelectric constants and dielectric constants, respectively. For simplicity, introduce the engineering notation of stress and strain as follows:

$$\sigma_1 = \sigma_{11},\ \sigma_2 = \sigma_{22},\ \sigma_3 = \sigma_{33},\ \sigma_4 = \sigma_{23},\ \sigma_5 = \sigma_{13},\ \sigma_6 = \sigma_{12}$$

$$\varepsilon_1 = \varepsilon_{11},\ \varepsilon_2 = \varepsilon_{22},\ \varepsilon_3 = \varepsilon_{33},\ \varepsilon_4 = 2\varepsilon_{23},\ \varepsilon_5 = 2\varepsilon_{13},\ \varepsilon_6 = 2\varepsilon_{12}$$

Then, (A1) can be rewritten in the matrix form as

$$\varepsilon = s\sigma + g^T D, \quad -E = g\sigma - \beta D \tag{A2}$$

For the generalized 2D problem, u_j and φ are dependent only on x_1 and x_2 such that

$$\varepsilon_3 = 0,\ E_3 = 0 \tag{A3}$$

Using Eq. (A2), Eq. (A3) gives

$$s_{31}\sigma_1 + \cdots + s_{36}\sigma_6 + g_{13}D_1 + g_{23}D_2 + g_{33}D_3 = 0 \tag{A4}$$

$$g_{31}\sigma_1 + \cdots + g_{36}\sigma_6 - \beta_{31}D_1 - \beta_{32}D_2 - \beta_{33}D_3 = 0 \tag{A5}$$

Solving σ_3 and D_3 from Eqs. (A4) and (A5), one obtains

$$\sigma_3 = -\frac{1}{s_{33}\beta_{33} + g_{33}^2}\left[\sum_{j=1,\neq 3}^{6}\left(\beta_{33}s_{3j} + g_{33}g_{3j}\right)\sigma_j + \sum_{k=1}^{2}\left(\beta_{33}g_{3k} - g_{33}\beta_{3k}\right)D_k\right] \tag{A6}$$

$$D_3 = -\frac{1}{s_{33}\beta_{33} + g_{33}^2}\left[\sum_{j=1,\neq 3}^{6}(g_{33}s_{3j} - s_{33}g_{3j})\sigma_j + \sum_{k=1}^{2}\left(s_{33}\beta_{3k} + g_{33}g_{3k}\right)D_k\right] \tag{A7}$$

Inserting Eqs. (A6) and (A7) into (A2), one has

$$\varepsilon_1 = \varepsilon_{11} = a_{11}\sigma_1 + a_{12}\sigma_2 + a_{14}\sigma_4 + a_{15}\sigma_5 + a_{16}\sigma_6 + b_{11}D_1 + b_{12}D_2 \tag{A8}$$

$$\varepsilon_5 = 2\varepsilon_{13} = a_{51}\sigma_1 + a_{52}\sigma_2 + a_{54}\sigma_4 + a_{55}\sigma_5 + a_{56}\sigma_6 + b_{15}D_1 + b_{25}D_2 \tag{A9}$$

$$-E_1 = b_{11}\sigma_1 + b_{12}\sigma_2 + b_{14}\sigma_4 + b_{15}\sigma_5 + b_{16}\sigma_6 - \delta_{11}D_1 - \delta_{12}D_2 \tag{A10}$$

where

$$a_{ij} = s_{ij} - \frac{s_{i3}s_{j3}\beta_{33} + s_{i3}g_{3j}g_{33} + s_{3j}g_{3i}g_{33} - s_{33}g_{3i}g_{3j}}{s_{33}\beta_{33} + g_{33}^2} \quad (i, j = 1, 2, 4, 5, 6)$$

$$b_{ij} = g_{ij} - \frac{s_{3j}g_{i3}\beta_{33} - s_{3j}g_{33}\beta_{i3} + g_{i3}g_{3j}g_{33} + s_{33}g_{3j}\beta_{i3}}{s_{33}\beta_{33} + g_{33}^2} \quad (i, j = 1, 2, 4, 5, 6)$$

$$\delta_{ij} = \beta_{ij} - \frac{g_{i3}g_{j3}\beta_{33} - g_{i3}\beta_{j3}g_{33} - \beta_{i3}g_{j3}g_{33} - s_{33}\beta_{i3}\beta_{j3}}{s_{33}\beta_{33} + g_{33}^2} \quad (i, j = 1, 2)$$

Mathematically, the upper and lower half-spaces come together at infinity, and therefore the following continuous conditions hold:

$$\left(\varepsilon_{11}^{\infty}\right)_1 = \left(\varepsilon_{11}^{\infty}\right)_2,\ \left(\varepsilon_{13}^{\infty}\right)_1 = \left(\varepsilon_{13}^{\infty}\right)_2,\ \left(E_1^{\infty}\right)_1 = \left(E_1^{\infty}\right)_2 \tag{A11}$$

Substituting Eqs. (A8)–(A10) into (A11), one can obtain

$$\begin{aligned}
&\begin{bmatrix} a_{11} & a_{15} & b_{11} \\ a_{15} & a_{55} & b_{51} \\ b_{11} & b_{51} & -\delta_{11} \end{bmatrix}_1 \begin{Bmatrix} \sigma_{11}^{\infty} \\ \sigma_{31}^{\infty} \\ D_1^{\infty} \end{Bmatrix}_1 - \begin{bmatrix} a_{11} & a_{15} & b_{11} \\ a_{15} & a_{55} & b_{51} \\ b_{11} & b_{51} & -\delta_{11} \end{bmatrix}_2 \begin{Bmatrix} \sigma_{11}^{\infty} \\ \sigma_{31}^{\infty} \\ D_1^{\infty} \end{Bmatrix}_2 \\
&= \left(\begin{bmatrix} a_{16} \\ a_{56} \\ b_{16} \end{bmatrix}_2 - \begin{bmatrix} a_{16} \\ a_{56} \\ b_{16} \end{bmatrix}_1 \right) \sigma_{21}^{\infty} + \left(\begin{bmatrix} a_{12} \\ a_{52} \\ b_{12} \end{bmatrix}_2 - \begin{bmatrix} a_{12} \\ a_{52} \\ b_{12} \end{bmatrix}_1 \right) \sigma_{22}^{\infty} \\
&+ \left(\begin{bmatrix} a_{14} \\ a_{54} \\ b_{14} \end{bmatrix}_2 - \begin{bmatrix} a_{14} \\ a_{54} \\ b_{14} \end{bmatrix}_1 \right) \sigma_{23}^{\infty} + \left(\begin{bmatrix} b_{12} \\ b_{52} \\ -\delta_{12} \end{bmatrix}_2 - \begin{bmatrix} b_{12} \\ b_{52} \\ -\delta_{12} \end{bmatrix}_1 \right) D_2^{\infty}
\end{aligned} \tag{A12}$$

which is the condition of loading at infinity.

If neglecting the terms related to electric variables, (A12) reduces

$$\begin{aligned}
&\begin{bmatrix} a_{11} & a_{15} \\ a_{15} & a_{55} \end{bmatrix}_1 \begin{Bmatrix} \sigma_{11}^{\infty} \\ \sigma_{31}^{\infty} \end{Bmatrix}_1 - \begin{bmatrix} a_{11} & a_{15} \\ a_{15} & a_{55} \end{bmatrix}_2 \begin{Bmatrix} \sigma_{11}^{\infty} \\ \sigma_{31}^{\infty} \end{Bmatrix}_2 \\
&= \left(\begin{bmatrix} a_{16} \\ a_{56} \end{bmatrix}_2 - \begin{bmatrix} a_{16} \\ a_{56} \end{bmatrix}_1 \right) \sigma_{21}^{\infty} + \left(\begin{bmatrix} a_{12} \\ a_{52} \end{bmatrix}_2 - \begin{bmatrix} a_{12} \\ a_{52} \end{bmatrix}_1 \right) \sigma_{22}^{\infty} + \left(\begin{bmatrix} a_{14} \\ a_{54} \end{bmatrix}_2 - \begin{bmatrix} a_{14} \\ a_{54} \end{bmatrix}_1 \right) \sigma_{23}^{\infty}
\end{aligned} \tag{A13}$$

which is consistent to the result of Chen and Hsu (1997). However, the derivation here is very explicit and concise.

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